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On the Orlicz function spaces $L_M(0,\infty)$.

Introduction.

It is a wellknown fact that the space $L_p(0,\infty)$ $1 \le p < \infty$ is isometric to $L_p(0,1)$ (even lattice isometric), and it is therefore natural to ask the question, whether the $L_p(0,\infty)$-spaces are the only symmetric function spaces on $(0,\infty)$, which are isomorphic to symmetric function spaces on $(0,1)$. Mityagyn [10] has conjectured that indeed it is so.

In this paper we investigate this conjecture for the class of symmetric function spaces consisting of the Orlicz function spaces $L_M(0,\infty)$, where M is an Orlicz function satisfying the Δ_2-condition.

In section 1 we investigate the isomorphic properties of the spaces $L_M(0,\infty)$; f.ex. we show that the set of Orlicz function N, for which the unit vector basis of the sequence space l_N is equivalent to a sequence of functions in $L_M(0,\infty)$ with mutually disjoint support consists exactly of these Orlicz functions, which up to equivalence belong to a natural compact convex subset of $C(0,1)$. This theorem is similar to the corresponding results for the spaces l_M and $L_M(0,1)$ proved by Lindenstrauss and Tzafriri [7],[8], and also the proof of it is close to theirs. We also show that the set of those p's $(1 \le p < \infty)$, for which the unit vector basis of l_p is equivalent to a sequence of functions in $L_M(0,\infty)$ with disjoint supports, is the union of two intervals, namely the two intervals $[\alpha_M, \beta_M]$ and $[\alpha_M^\infty, \beta_M^\infty]$ associated to respectively l_M and $L_M(0,1)$ in [8]. Finally we modify the "A_M^ε - technique" of Kadec and Pełczyński [3] to get a technique, which can be applied to function spaces over sets of infinite measure.

In section 2 we present our main results, related to the conjecture above. Through a series of propositions and theorems we show that if an reflexive Orlicz space $L_M(o,\infty)$ with the intervals $[\alpha_M,\beta_M]$ and $[\alpha_M^\infty,\beta_M^\infty]$ to the "same side of 2" is isomorphic to a symmetric function space on $[0,1]$, then $L_M(0,\infty)$ is isomorphic to $L_M(0,1)$ and $[\alpha_M,\beta_M] \subseteq [\alpha_M^\infty,\beta_M^\infty]$. The same is true if the condition above is loosened to $2 \notin [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty]$ and the above symmetric function space is an Orlicz space. These results give of course a lot of examples of Orlicz function spaces on $(0,\infty)$ which are not isomorphic to any symmetric function space on $[0,1]$ (namely spaces, where the two intervals are disjoint.)

The results above are then used to show one of the main results in the section, namely an affirmative answer to the above conjecture for the Orlicz spaces $L_M(0,\infty)$, where M is an Orlicz function which is slowly varying at ∞ (i.e. $\lim\limits_{t\to\infty} \frac{M(tx)}{M(t)}$ exists for all $x \in [0,1]$), and whose intervals satisfy the above conditions.

Finally we present some general results, which indicate, how the conjecture might be proved for general Orlicz spaces.

0 Preliminaries.

In this paper we shall use the standard notation of the theory of Banach spaces, as it appears in [5]; let us just here recall that if $\bar{X}$ and Y are Banach spaces, then the Banach distance $d(\bar{X},Y)$ between $\bar{X}$ and Y is defined by

$$d(\bar{X},Y) = \inf\{\|T\|\,\|T^{-1}\| \mid T \text{ isomorphism of } \bar{X} \text{ onto } Y\},$$

if $\bar{X}$ and Y are isomorphic and ∞ else.

By an Orlicz function we shall always mean a continuous convex non-decreasing function $M: [0,\infty[\to [0,\infty[$, so that $M(0) = 0$, $M(1) = 1$ and $M(x) > 0$ for $x > 0$.

The Lesbeque measure on $[0,\infty[$ will in this work always be denoted by the letter λ.

Let M be an Orlicz function and let $A \subseteq [0,\infty[$ be a λ - measurable set. The Orlicz function space $L_M(A)$ consists of those measurable functions f defined on A, for which there exists an r, so that $I_r(f) = \int_A M(r^{-1}|f|)d\lambda < \infty$.

With the norm $\|f\| = \inf\{r \mid I_r(f) \leq 1\}$ $L_M(A)$ is a Banach space.

Similarly the Orlicz sequence space l_M consists of all those sequences (t_n) of scalars for which there is an $r > 0$ with

$$I_r((t_n)) = \sum_{n=1}^{\infty} M(r^{-1}|t_n|) < \infty .$$

With the norm $\|(t_n)\| = \inf\{r \mid I_r((t_n)) < \infty\}$ l_M is a Banach space. For the basic properties of Orlicz spaces we refer to [9]. In this paper we shall only work with separable Orlicz spaces l_M, $L_M(0,1)$ and $L_M(0,\infty)$, and therefore we shall always assume (unless otherwise stated) that the Orlicz function M satisfies the Δ_2-condition; i.e. there is a constant $K \geq 1$ so that

(*) $\quad M(2x) \leq KM(x) \qquad x \geq 0$.

The smallest constant which can be used in this inequality is called the Δ_2-constant of M.

Here we should keep in mind that when we consider the space l_M only the values of M close to 0 are important, and for the space $L_M(0,1)$ only the values close to ∞ are important, while for $L_M(0,\infty)$ the values of M both close to 0 and to ∞ matters. (See f.ex [9]). Therefore when we consider the sequence space l_M we shall often consider M as an element of $C(0,1)$.

The sequence $(e_n) \subseteq l_M$, where $e_n = (\delta_n^i)_i$, is called the unit vector basis of l_M; when M satisfies the Δ_2-condition (equivalently when l_M is separable) (e_n) is a symmetric basis in l_M.

If M and N are Orlicz functions, then we say that M and N are equivalent, if there is a constant $K \geq 1$, so that

$$(**). \quad K^{-1} N(x) \leq M(x) \leq KN(x) \qquad x \geq 0 .$$

We shall say that M and N are equivalent at 0 (respectively at ∞), if (**) holds in a neighbourhood of 0 (respectively ∞).

For a detailed study of the isomorphic properties of the spaces l_M and $L_M(0,1)$ we refer to [4], [6],[7] and [8]; see also [5].

If M is an Orlicz function, then we shall make use of the following important sets related to M :

$$E_M^\infty = \{N \in C(0,1) \mid \exists (t_n) \subseteq R \quad t_n \to \infty \text{ with } N(x) = \lim_n \frac{M(t_n x)}{M(t_n)}\} .$$

$$C_M^\infty = \overline{\text{conv}}\, E_M^\infty \qquad \text{(closure taken i} \;\; C(0,1).)$$

For $o < s < 1$ we define

$$E_{M,s} = \{N \in C(0,1) \mid \exists t \quad 0 < t \leq S \quad \text{with} \quad M(x) = \frac{M(tx)}{M(t)}\}$$

$$C_{M,s} = \overline{\text{conv}}\, E_{M,s} .$$

$$E_M = \bigcap_{0<s\leq 1} E_{M,s} .$$

$$C_M = \overline{\text{conv}}\, E_M .$$

It follows from the Δ_2-condition that all the above sets are compact subsets of $C(0,1)$.

It was proved in [7], that the unit vector basis of an Orlicz sequence space l_N is equivalent to a block basis sequence of the unit vector basis of l_M if and only if $N \in C_{M,1}$, and in [8] it is shown that the unit vector basis of l_N is equivalent to a sequence of functions in $L_M(0,1)$ with mutually disjoint supports if and only if $N \in C_M^\infty$.

If M is an Orlicz function then the following numbers are important

$$\alpha_M = \sup\{p \mid \sup_{0<x,t\leq 1} \frac{M(tx)}{M(t)x^P} < \infty\}$$

$$\beta_M = \inf\{p \mid \inf_{0<x,t\leq 1} \frac{M(tx)}{M(t)x^P} > 0\}$$

$$\alpha_M^\infty = \sup\{p \mid \sup_{x,y\geq 1} \frac{M(x)y^P}{M(xy)} < \infty\}$$

$$\beta_M^\infty = \inf\{p \mid \inf_{x,y\geq 1} \frac{M(x)y^P}{M(xy)} > 0\} .$$

It is proved in [8], that both intervals $[\alpha_M,\beta_M]$ and $[\alpha_M^\infty,\beta_M^\infty]$ are compact subsets of $[1,\infty[$ and that $p \in [\alpha_M,\beta_M]$ if and only if the unit vector basis of l_p is equivalent to a block basis sequence of the unit vector basis of l_M, while $p \in [\alpha_M^\infty,\beta_M^\infty]$ if and only if the unit vector basis of l_p is equivalent to a sequence of functions in $L_M(0,1)$ with mutually disjoint supports.

1. The isomorphic properties of the spaces $L_M(0,\infty)$.

In this section let M be a fixed Orlicz function and define the sets:

$$E_M(0,\infty) = \overline{\{N \in C(0,1) \mid \exists t > 0 \text{ with } N(x) = \frac{M(tx)}{M(t)}\ 0 \leq x \leq 1\}} .$$

$$C_M(0,\infty) = \overline{\operatorname{conv}}\, E_M\ (0,\infty) .$$

It follows immediately from the Δ_2-condition of M that both $E_M(0,\infty)$ and $C_M(0,\infty)$ are compact subsets of $C(0,1)$.

Our first theorem shows that the set $C_M(0,\infty)$ plays the same role for $L_M(0,\infty)$ as $C_{M,1}$ and C_M^∞ do for the spaces l_M and $L_M(0,1)$ respectively, the proof of it is similar to the corresponding results in [7] and [8] .

1.1 Theorem

(i) Every normalized sequence of mutually disjoint[x)] elements from $L_M(0,\infty)$ has a subsequence, which is equivalent to the unit vector basis of an Orlicz sequence space l_N for some $N \in C_M(0,\infty)$.

(ii) If N is an Orlicz function, the unit vector basis of l_N is equivalent to a sequence of mutually disjoint elements of $L_M(0,\infty)$ if and only if N is equivalent at 0 to a function in $C_M(0,\infty)$.

Proof

(i): Let $(f_n) \subseteq L_M(0,\infty)$, with $\|f_n\| = 1$ and $f_n \perp f_m$ for $n \neq m$ and put $A_n = \operatorname{supp} f_n$.

For each $n \in \mathbb{N}$ we define the Orlicz function

$$(1) \qquad M_n(x) = \int_0^\infty M(x|f_n(t)|)dt = \int_{A_n} \frac{M(x|f_n(t)|)}{M(|f_n(t)|)} d\mu(t)$$

where μ is the measure with $\frac{d\mu}{dt} = M(|f_n|)$.

Since $\mu(0,\infty) = \mu(A_n) = 1$ it follows immediately from (1) that $M_n \in C_M(0,\infty)$.

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x) Two elements $f,g \in L_M(0,\infty)$ are said to be disjoint, if they have disjoint supports, and in that case we write $f \perp g$.

From the compactness of $C_M(0,\infty)$ it follows that there is a subsequence (M_{n_j}) of (M_n) and a $N \in C_M(0,\infty)$ so that

(2) $|M_{n_j}(x) - N(x)| < 2^{-j}$ for all $x \in [0,1]$, all $j \in \mathbb{N}$.

Since $\sum_j t_j f_{n_j}$ converges if and only if $\sum_j M_{n_j}(|t_j|) < \infty$

it follows from (2) that (f_{n_j}) is equivalent to the unit vector basis of l_N. This finishes the proof of (i).

(ii): Let $N \in C_M(0,\infty)$.. By the Krein- Milman theorem we get that there is a probalility measure μ on $E_M(0,\infty)$ so that

(3) $N(x) = \int_{E_M(0,\infty)} G(x)d\mu(G) \qquad x \in [0,1]$.

If we put $\alpha_1 = \mu(E_{M,1})$, $\alpha_2 = \mu(E_M^\infty \setminus E_{M,1})$ and $\alpha_3 = 1 - \alpha_1 - \alpha_2$, then it follows from (3) that N can be written as

(4) $N = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3$.

where $N_1 \in C_{M,1}$, $N_2 \in C_M^\infty$ and

(5) $N_3(x) = \frac{1}{\alpha_3} \int_{E_M(0,\infty)\setminus E_{M,1} \cup E_M^\infty} G(x)d\mu(G)$.

(If for an i $\alpha_i = 0$ N_i is not occuring in (4)).

Let us now divide $[0,\infty[$ into three disjoint measurable sets A_1, A_2 and A_3 with $\lambda(A_i) = \infty$ $i = 1,3$ and $\lambda(A_2) = 1$.

If (B_n) is a sequence of disjoint measurable subsets of A_1 with $\lambda(B_n) = 1$ then it is readily seen that the sequence (1_{B_n}) is equivalent to the unit vector basis of l_M, and since [7] gives that the unit vector basis of l_{N_1} is equivalent to a block basis sequence of l_M, it follows that there is a sequence $(f_n) \subseteq L_M(0,\infty)$ with $f_n \perp f_m$ $n \neq m$ and $\mathrm{supp}(f_n) \subseteq A_1$ for all n, so that (f_n) is equivalent to the unit vector basis of l_{N_1}. Since by [8] the unit vector basis of l_{N_2} is equivalent to a sequence of mutually disjoint elements from $L_M(0,1)$ it follows immediately that there is a sequence $(g_n) \subseteq L_M(0,\infty)$ of mutually disjoint elements equivalent to the vector basis of l_{N_2} and so that $\mathrm{supp}(g_n) \subseteq A_2$ for all n.

It is easy to see from the form of N_3 that there is a probability measure ν on $[1,\infty[$ so that $\nu(\{1\}) = 0$ and so that

$$(5) \qquad N_3(x) = \int_1^\infty \frac{M(tx)}{M(t)}\, d\nu(t) . \qquad x \in [0,1] .$$

The Δ_2-condition on M gives together with (5) that N_3 is equivalent to N_4 given by

$$(6) \qquad N_4(x) = \sum_{k=0}^{\infty} \nu_k \frac{M(2^k x)}{M(2^k)} . \qquad x \in [0,1]$$

where $\nu_k = \nu([2^k, 2^{k+1}[)$.

Let (C_n) be a sequence of mutually disjoint measurable subsets of A_3 with

$$\lambda(C_n) = \sum_{k=0}^{\infty} \frac{\nu_k}{M(2^k)} \qquad \text{for all } n$$

and let for each n $(C_{n,k})$ be a disjoint partition of C_n into measurable sets with $\lambda(C_{n,k}) = \frac{\nu_k}{M(2^k)}$.

For each n we define

$$h_n = \sum_{k=0}^{\infty} 2^k . 1_{C_{n,k}} \qquad \text{(convergence in } L_M(0,\infty)).$$

It is now trivial to check that (h_n) is equivalent to the unit vector basis of l_{N_4} (and l_{N_3}).

Define

$$r_n = f_n + g_n + h_n.$$

Using (4) it easily follows that (r_n) is equivalent to the unit vector basis of l_N.

The other implication in (ii) follows directly from (i).

q.e.d.

If $(f_n) \subseteq L_M(0,\infty)$ then it often of interest to investigate, whether or not (f_n) has a subsequence, which is equivalent to a sequence of mutually disjoint functions. A method to do this is given by Kadec-Pełczyński [3] (see also [8] in case of the space $L_M(0,1)$ (or for that matter $L_M(A)$, where A has finite measure). This method is not directly appliable when we work with Orlicz function spaces over sets with infinite measure, but it has to be combined with another technique, which we are going to explain now:

1.2 Definition

A subset $\underline{X} \subseteq L_M(0,\infty)$ is said to satisfy condition (A), if the following holds:

(A) $\quad \forall \varepsilon > 0 \quad \forall m \in \mathbb{N} \quad \exists f \in \underline{X}$ so that

$$\| f \cdot 1_{[0,m]} \| < \varepsilon \| f \| .$$

If $B \subseteq [0,\infty[$ is a set of finite measure and $\varepsilon > 0$, then we define the set $A_M^{\varepsilon}(B)$ to be:

$$A_M^{\varepsilon}(B) = \{ f \in L_M(0,\infty) \mid \operatorname{supp} f \subseteq B \text{ and }$$

$$\lambda\{ t \in B \mid |f(t)| > \varepsilon \} > \varepsilon \}.$$

We are now able to show:

<u>1.3 Proposition</u>

Let $\underline{X} \subseteq L_M(0,\infty)$ be a subset satisfying condition (A). Then there is a sequence $(f_n) \subseteq \underline{X}$, which is equivalent to a sequence of mutually disjoint elements of $L_M(0,\infty)$.

<u>Proof</u>

Since $\underline{X}$ satisfies condition (A) we can by induction construct sequences $(m_n) \subseteq \mathbb{N}$ and $(f_n) \subseteq \underline{X}$, $f_n \neq 0$ for all n, so that

(1) $\quad 4 \| f_n \cdot 1_{[m_n,\infty[} \| < 2^{-n-2} \| f_n \| \qquad n = 1,2,\ldots$

(2) $\quad 4 \| f_{n+1} \cdot 1_{[0,m_n]} \| < 2^{-n-2} \| f_n \| \qquad n = 1,2,\ldots$

If we define $g_n = \| f_n \|^{-1} f_n$ and $h_n = g_{n+1} \cdot 1_{]m_n,m_{n+1}]}$ for all $n \in \mathbb{N}$, then we have:

$$1 = \| g_{n+1} \| \leq \| g_{n+1} \cdot 1_{[0,m_n]} \| + \| h_n \| + \| g_{n+1} \cdot 1_{]m_{n+1},\infty[} \|$$

$$\leq 2^{-n-3} + \| h_n \| \qquad \text{for all } n \in \mathbb{N}.$$

and hence $\| h_n \| \geq 2^{-1}$ for all n.

For the sequence $(h_n^*) \subseteq [h_n]^*$ biorthogonal to the basic sequence (h_n) we now get:

(3) $\quad ||h_n^*|| \leq 2||h_n||^{-1} \leq 4 \quad$ for all n

and hence

(4) $$||h_n^*||\ ||g_{n+1} - h_n|| \leq 4[||g_{n+1}\cdot 1_{[0,m_n]}|| + ||g_{n+1}\cdot 1_{]m_{n+1}\infty[}||] \leq 2^{-n-1}$$

and from this we obtain:

(5) $$\sum_{n=1}^{\infty} ||h_n^*||\ ||g_{n+1} - h_n|| \leq 2^{-1}$$

By the stability theorems of Bessaga and Pełczyński [1] (5) implies that (g_{n+1}) is a basic sequence equivalent to (h_n) . This proves the proposition.

1.4 Corollary

If $\bar{X}$ is a subspace of $L_M(0,\infty)$, then either there is a normalized sequence $(f_n) \subseteq \bar{X}$, which is equivalent to the unit vector basis of l_N for some $N \in C_M(0,\infty)$, or there is an $\varepsilon > 0$, a $B \subseteq [0,\infty[$ with $0 < \lambda(B) < \infty$ and a subspace Y of $A_M^{\varepsilon}(B)$, so that $\bar{X}$ is isomorphic to Y .

Proof

Suppose that for no $N \in C_M(0,\infty)$ there is a sequence in $\bar{X}$ equivalent to the unit vector basis of l_N . Then by theorem 1.1 and proposition 1.3 $\bar{X}$ does not satisfy condition (A) . Hence then there is a $\delta > 0$ and an m so that

(1) $\quad ||f\cdot 1_{[0,m]}|| \geq \delta||f|| \quad$ for all $f \in \bar{X}$.

Put $B = [0,m]$ and let P be the projection in $L_M(0,\infty)$ defined by $Pf = f\cdot 1_B$ for all $f \in L_M(0,\infty)$ and put $Y = P(\bar{X})$.

(1) shows that $P_{|\bar{X}}$ is an isomorphism of $\bar{X}$ onto Y . From our assumption on $\bar{X}$ and proposition 3 of [8] it now follows that there is an $\varepsilon > 0$, so that $Y \subseteq A_M^{\varepsilon}(B)$.

q.e.d.

Remark.

Using a refined version of the argument in proposition 1.3 it is easy to see that if (f_n) is a basic sequence in $L_M(0,\infty)$, so that $[f_n]$ satisfies condition (A), then there is a block basic sequence with respect to (f_n), which is equivalent to the unit vector basis of l_N for some $N \in C_M(0,\infty)$.

We now turn our attention to the set of p's for which x^p is equivalent at 0 to a function $N \in C_M(0,\infty)$. We have the following theorem.

1.5 Theorem.

The following statements are equivalent:

(i) $p \in [\alpha_M, \beta_M] \cup [\alpha_M^{\infty}, \beta_M^{\infty}]$.

(ii) $x^p \in C_M \cup C_M^{\infty}$.

(iii) x^p is equivalent at 0 to a function in $C_M(0,\infty)$.

Proof.

It follows from [8] that (i) is equivalent to (ii) and trivially (ii) $\Rightarrow$ (iii) , so we have left to prove f.ex (iii) $\Rightarrow$ (i) .

Let us first note that (iii) actually implies that $x^p \in C_M(0,\infty)$; indeed, using the fixpoint procedure of [6] on an element $N_p \in C_M(0,\infty)$ with $x^p \sim N_p$ we get that there is a $q \geq 1$ with $x^q \in C_{N_p} \subseteq C_M(0,\infty)$, but then obviously $p = q$.

Hence the implication (iii) $\Rightarrow$ (i) will follow, if we prove

(1) $p \notin [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty] \Rightarrow x^p \notin C_M(0,\infty)$.

Suppose first that $p < \min(\alpha_M,\alpha_M^\infty)$ and let q be chosen so that $p < q < \min(\alpha_M,\alpha_M^\infty)$. Then there is a constant C so that

(1) $$\frac{M(tx)}{M(t)} \leq Cx^q \quad x \in [0,1] \ , \ t \in]0,1]$$

(2) $$\frac{M(t)x^q}{M(tx)} \leq C \quad x \geq 1 \quad t \geq 1 \ .$$

From (1) and (2) we conclude that

(3) $$\frac{M(tx)}{M(t)} \leq C^2x^q \quad \text{for all} \quad x \in [0,1] \quad \text{and all} \quad t > 0 \ .$$

(3) gives immediately that if $N \in C_M(0,\infty)$ then

(4) $N(x) \leq C^2x^q$ for $x \in [0,1]$

and hence $x^p \notin C_M(0,\infty)$.

The case where $p > \max(\beta_M,\beta_M^\infty)$ is treated in the same way.

Suppose next that $\beta_M < p < \alpha_M^\infty$, and let us choose q_1 and q_2 so that $\beta_M < q_1 < p < q_2 < \alpha_M^\infty$.

There exists a constant $C > 0$ so that

(5) $$\frac{M(tx)}{M(t)} \geq C^{-1}x^{q_1} \quad x \in [0,1] \quad t \in]0,1]$$

(6) $$\frac{M(t)x^{q_2}}{M(tx)} \leq C \quad x \geq 1 \quad t \geq 1 \ .$$

Let $N \in C_M(0,\infty)$ be of the form

$$(7) \qquad N(x) = \int_1^\infty \frac{M(tx)}{M(t)} d\mu(t) \qquad x \in [0,1]$$

where μ is a probability measure on $[1,\infty[$ with $\mu\{1\} = 0$, and let $K \subseteq [1,\infty[$ be a compact set with $\mu(K) > 0$. Since M satisfies the Δ_2-condition, there is a constant $C_1 > 0$, so that

$$(8) \qquad N(x) = \int_1^\infty \frac{M(tx)}{M(t)} d\mu(t) \geq \int_K \frac{M(tx)}{M(t)} d\mu(t) \geq C_1 M(x) \geq C_1 C^{-1} x^{q_1} \qquad x \in [0,1] .$$

If now $F \in C_M(0,1)$ is arbitrary, then we can write F as a convex combination:

$$(9) \qquad F(x) = \alpha_1 N_1(x) + \alpha_2 N_2(x) + \alpha_3 N_3(x) \qquad x \in [0,1]$$

where $N_1 \in C_{M,1}$, $N_2 \in C_M^\infty$ and N_3 is of the form (7) .
From (5) and (8) it now follows ~~that~~ if either $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$, then there is a constant C_2 , so that

$$(10) \qquad F(x) \geq C_2 x^{q_1} \qquad x \in [0,1]$$

and if $F \in C_M^\infty$, then it follows from (6) that

$$(11) \qquad F(x) \leq C x^{q_2} .$$

(10) and (11) give that $p \notin C_M(0,\infty)$.

The case $\beta_M^\infty < p < \alpha_M$ is treated in the same way.

<u>1.6 Corollary</u>

Let M be an Orlicz function, so that $\alpha_M^\infty \geq 2$.

If l_p is isomorphic to a subspace of $L_M(0,\infty)$ then either

$p \in [\alpha_M, \beta_M] \cup [\alpha_M^\infty, \beta_M^\infty]$ or $p = 2$.

Proof

Let $(f_n) \subsetneq L_M(0,\infty)$ be a normalized sequence equivalent to the unit vector basis of l_p . By corollary 1.4 and the remark just after either there is a normalized block basis sequence (g_n) of (f_n) , which is equivalent to a sequence of mutually disjoint functions of $L_M(0,\infty)$ or $[f_n]$ is isomorphic to a subspace of $A_M^{\varepsilon}(B)$ for some B with $0 < \lambda(B) < \infty$. In the first case we get $x^P \in C_M(0,\infty)$, since (g_n) is equivalent to (f_n) and in the second case it follows from [8] corollary 4 that $p = 2$.

q.e.d.

The problem of Orlicz spaces $L_M(0,\infty)$ being isomorphic to symmetric function spaces on $[0,1]$.

We recall that a Banach space $L\rho(0,1)$ with norm ρ of Lesbegue measurable functions on $[0,1]$ is called a function space, if the following conditions are satisfied

(i) $L\rho(0,1)$ is a Banach lattice in the usual ordering of the measurable functions.

(ii) All indicator functions of measurable sets belong to $L_\rho(0,1)$.

(iii) $(f_n) \subseteq L_\rho(0,1)$ $f_n \uparrow f$ a.e with $(\rho(f_n))$ bounded $\Longrightarrow f \in L_\rho(0,1)$ and $\rho(f_n) \uparrow \rho(f)$.

(iv) There is a constant $K > 0$ so that $\int_0^1 |f| d\lambda \leq K\rho(f)$ for all $f \in L_\rho(0,1)$.

Since we only consider the separable case here, we shall also assume (see [9]) .

(v) If (E_n) a sequence of measurable sets with $\lambda(E_n) \to 0$, then $\rho(f \cdot 1_{E_n}) \to 0$ for all $f \in L_\rho(0,1)$.

We shall say that a function space $L_\rho(0,1)$ is symmetric provided.

(vi) For all measure preserving 1-1 maps φ of $[0,1]$ onto $[0,1]$, $f \in L_\rho(0,1)$ implies $f \circ \varphi \in L_\rho(0,1)$.

Our first lemma is an easy consequence of a result of Bretagnolle and Dacunha - Castelle [2].

2.1 Lemma.

If $L_M(0,\infty)$ is a reflexive Orlicz space with $\max(\beta_M, \beta_M^\infty) < 2$, then $L_M(0,\infty)$ can be imbedded isomorphically into $L_p(0,1)$ for every p, $1 \leq p < \min(\alpha_M, \alpha_M^\infty)$.

Proof.

Let $1 \leq p < \min(\alpha_M, \alpha_M^\infty)$. By going to an Orlicz function equivalent to M if neccesary we may assume that

$$p < \liminf_{x \to 0} \frac{xM'(x)}{M(x)}, \quad p < \liminf_{x \to \infty} \frac{xM'(x)}{M(x)},$$

$\limsup\limits_{x \to 0} \dfrac{xM'(x)}{M(x)} < 2$ and $\limsup\limits_{x \to \infty} \dfrac{xM'(x)}{M(x)} < 2$ (see f.ex [8]), from this it follows that there are neighbourhoods U_0 and U_∞ of 0 and ∞ respectively so that in U_0 $M(x)x^{-P}$ is increasing and in U_∞ $M(x)x^{-2}$ is decreasing, and therefore by [2], theorem IV 1, $L_M(0,\infty)$ can be imbedded into $L_p(0,1)$.

q.e.d.

2.2 Corollary.

Suppose that either $2 < \min(\alpha_M, \alpha_M^\infty)$ or $\max(\beta_M, \beta_M^\infty) < 2$ and that $L_M(0,\infty)$ is reflexive. If $L_M(0,\infty)$ is isomorphic to a symmetric function space $L_\rho(0,1)$, then there is an Orlicz function N so that ρ is equivalent to the Orlicz norm determined by N (that is $L_N(o,1)$ and $L_\rho(0,1)$ are isomorphic via the identity map).

Proof.

Suppose that $\max(\beta_M, \beta_M^\infty) < 2$. By our assumption and lemma 2.1 $L_\rho(0,1)$ can be imbedded isomorphically into $L_1(0,1)$, and by a result of Dacunha - Castelle [2], this implies that there is an Orlicz function N with the properties stated. The case $\min(\alpha_M, \alpha_M^\infty) > 2$ is obtained by duality.

q.e.d.

We are going to show that the Orlicz function N appearing in corollary 2.2 is actually equivalent to M at ∞. This will follow from the following theorem, where the idea of proof is essentially the same as in the proof of the corresponding result by Lindenstrauss and Tzafriri [8], th. 4 in case of Orlicz spaces on $[o,1]$.

2.3 Theorem.

Let M be an Orlicz function with $\max(\beta_M, \beta_M^\infty) < 2$. If N is an Orlicz function so that $L_N(0,1)$ is isomorphic to a subspace of $L_M(0,\infty)$, then

$$\sup_{x \geq 1} \frac{M(x)}{N(x)} < \infty .$$

<u>Proof</u>.

Let $T: L_N(0,1) \to L_M(0,\infty)$ denote an isomorphism, and put for $m \in \mathbb{N}$ and $1 \leq i \leq m$ $\varphi_{i,m} = 1_{[i-1/m,i/m]}$.

$$\gamma_m = \|\varphi_{i,m}\|_N \quad \text{and} \quad f_{i,m} = \gamma_m^{-1} T\varphi_{i,m} .$$

From the fact that $\|T^{-1}\|^{-1} \leq \|f_{i,m}\|_M \leq \|T\|$ for all i and m together with the Δ_2-property of M we get that there is a constant $C \geq 1$ so that

$$(1) \qquad C^{-1} \leq \int_0^\infty M(|f_{i,m}(t)|dt \leq C \qquad m \in \mathbb{N}\ 1 \leq i \leq m .$$

It follows immediately from the definition of the $f_{i,m}$'s that for all possible choices of signs $\theta_i(= \pm 1)$ $i = 1.2..m.$ we have

$$(2) \qquad \gamma_M^{-1}\|T^{-1}\|^{-1} \leq \|\sum_{i=1}^m \theta_i f_{i,m}\|_M \leq \gamma_m^{-1}\|T\|$$

By [2] , page 470 we get that there is a constant K_1 only dependent on M , so that

$$(3) \qquad E\,(\int_0^\infty M(|\sum_{i=1}^m \theta_i f_{i,m}(t)|)dt \leq K_1 \sum_{i=1}^m \int_0^\infty M(|f_{i,m}(t)|)dt .$$

where E denotes the average of all 2^m possible signs.

Hence for at least half the choices of signs we have

$$(4) \qquad \int_0^\infty M(|\sum_{i=1}^m \theta_i f_{i,m}(t)|)dt \leq 2K \sum_{i=1}^m \int_0^\infty M(|f_{i,m}(t)|)dt .$$

As in the proof of theorem 4 of [8] , we can now using (4) inductively for $m_n = 2^n$ $n \in \mathbb{N}$ choose signs $(\theta_i^{m_n})_{i=1}^{m_n}$, so that (4) is satisfied and so that the functions

$$(5)\qquad \psi_n = \sum_{i=1}^{m_n} \theta_i^{m_n} \varphi_{i,m_n} \qquad n \in \mathbb{N} \quad 1 \le i \le m_n$$

are asymptotically orthogonal (i.e. that for all k, there is an $n(k)$ with $|\int_0^1 \psi_j(t)\psi_n(t)| \le 2^{-k}$ $1 \le j \le k$, $n \ge n_k$) ; hence by the Khinchin inequality (ψ_{n_j}) is equivalent to the unit vector basis of l_2 for every subsequence (n_j) tending sufficiently fast to ∞. Since by assumption the unit vector basis of l_2 is not sitting on disjoint blocks in $L_M(0,\infty)$ we get from this together with proposition 1.3 that there is a measurable set $A \subseteq [0,\infty[$ with $0 < \lambda(A) < \infty$ and a $\delta > 0$ so that

$$(6)\qquad \delta\|h_n\|_M \le \|h_n \cdot 1_A\|_M$$

where $h_n = \gamma_{m_n}^{-1} T\psi_n \qquad n \in \mathbb{N}$.

We now claim that there is an $\varepsilon > 0$ so that

$$(7)\qquad \lambda(\{t \in A \mid |h_n(t)| \ge \varepsilon\|h_n \cdot 1_A\|_M\}) \ge \varepsilon .$$

Indeed, if not, then by the above and proposition 3 of [8] there is a subsequence (n_j) so that (ψ_{n_j}) is equivalent to the unit vector basis of l_2 and so that $(T\psi_{n_j} \cdot 1_A)$ is equivalent to the unit vector basis of l_F for some $F \in C_M^\infty$. The first statement gives: $\sum_{j=1}^\infty |t_j|^2 < \infty \Rightarrow \sum_{j=1}^\infty t_j T\psi_{n_j} \cdot 1_A$ is convergent, and since by assumption there is a constant B, so that $x^2 \le BF(x)$ $0 \le x \le 1$ the second one gives that $\sum_{j=1}^\infty t_j T\psi_{n_j} \cdot 1_A$ convergent $\Rightarrow \sum_{j=1}^\infty |t_j|^2 < \infty$, and hence F is equivalent to x^2, which is a contradiction.

From (1), (4), (6) and (7) we now obtain.

$$(8) \qquad \varepsilon M(\varepsilon\delta\|h_n\|) \le \int_0^\infty M(|h_n(t)|)dt \le \int_0^\infty M(\gamma_{2^n}^{-1}|(T\psi_n)(t)|)dt \le$$

$$2K_1 \sum_{i=1}^{2^n} \int_0^\infty M(|f_{i,2^n}(t)|)dt \le 2CK_1 \cdot 2^n \quad \text{all } n .$$

which gives

$$(9) \qquad \varepsilon M(\gamma_{2^n}^{-1}\|T^{-1}\|^{-1}\varepsilon\delta) \le 2K_1 C \cdot 2^n . \quad \text{for all } n .$$

By the Δ_2-property of M (9) gives that there is a constant K, so that

$$(10) \qquad M(\gamma_{2^n}^{-1}) \le K2^n \quad \text{all } n .$$

and hence if $\gamma_{2^n}^{-1} \le x \le \gamma_{2^{n+1}}^{-1}$

$$M(x) \le M(\gamma_{2^{n+1}}^{-1}) \le K \cdot 2^{n+1} = 2KN(\gamma_{2^n}^{-1}) \le 2KN(x)$$

This concludes the proof of theorem.

We can now show:

<u>2.4 Theorem</u>.

Let $L_M(0,\infty)$ be a reflexive Orlicz space so that $2 \notin [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty]$. Then:

(i) If $L_M(0,\infty)$ is isomorphic to a symmetric function space $L_\rho(0,1)$ and either $\min(\alpha_M,\alpha_M^\infty) > 2$ or $\max(\beta_M,\beta_M^\infty) < 2$ then $L_\rho(0,1)$ is isomorphic to $L_M(0,1)$.

(ii) If $L_M(0,\infty)$ is isomorphic to an Orlicz space $L_N(0,1)$, then M is equivalent to N at ∞ and $[\alpha_M,\beta_M] \subseteq [\alpha_M^\infty,\beta_M^\infty]$.

Proof.

By Corollary 2,2 (i) is a special case of (ii), so we have to prove the latter.

Let M^* and N^* denote the conjugate Orlicz functions to M, respectively N.

Let us assume that $\beta_M^\infty < 2$.

If $p \in [\alpha_{N^*}^\infty, \beta_{N^*}^\infty]$ then since $\alpha_{M^*}^\infty > 2$ we get that either $p = 2$ or $p \in [\alpha_{M^*}, \beta_{M^*}] \cup [\alpha_{M^*}^\infty, \beta_{M^*}^\infty]$.

This gives us that either

(1) $$[\alpha_N^\infty, \beta_N^\infty] \subseteq [\alpha_M, \beta_M] \cup [\alpha_M^\infty, \beta_M^\infty]$$

or else $\alpha_N^\infty = \beta_N^\infty = 2$, but since $L_N(0,1)$ contains subspaces isomorphic to l_p for $p \neq 2$ by our assumption this is excluded by Corollary 4 of [8]. Hence (1) holds.

If $\beta_M > 2$, then $[\alpha_M, \beta_M] \cap [\alpha_M^\infty, \beta_M^\infty] = \emptyset$ and hence either

(2) $$[\alpha_N^\infty, \beta_N^\infty] \subseteq [\alpha_M^\infty, \beta_M^\infty]$$

or

(3) $$[\alpha_N^\infty, \beta_N^\infty] \subseteq [\alpha_M, \beta_M].$$

If (2) occurs then $\alpha_{N^*}^\infty > 2$ and since $L_{M^*}(0,\infty)$ is isomorphic to $L_{N^*}(0,1)$, $L_{N^*}(0,1)$ contains isomorphs of l_p for each $p \in [\alpha_{M^*}, \beta_{M^*}]$ and hence by the corollary cited above $[\alpha_{M^*}\ \beta_{M^*}] \subseteq [\alpha_{N^*}^\infty, \beta_{N^*}^\infty]$, which is a contradiction; similarly (3) leads to a contradiction, and therefore we must have $\beta_M < 2$. Using now theorem 2.3 we obtain

(4) $$\sup_{x \geq 1} \frac{M(x)}{N(x)} < \infty .$$

From (1) we obtain that $\beta_N^\infty < 2$ and therefore (since $L_M(0,1)$ can be imbedded into $L_N(0,1)$) we get from theorem 4 of [8] that

(5) $$\sup_{x \geq 1} \frac{N(x)}{M(x)} < \infty .$$

Arguments similar to the ones above also show that $[\alpha_M, \beta_M] \subseteq [\alpha_N^\infty, \beta_N^\infty]$, but by (4) and (5) $\alpha_N^\infty = \alpha_M^\infty$ and $\beta_M^\infty = \beta_N^\infty$.

The case, where $\alpha_M^\infty > 2$ follows from the already shown by dualization.

q.e.d.

Let us recall that if N is an Orlicz function and $F(x) = \lim_{t\to\infty} \frac{N(tx)}{N(t)}$ exists for every $x \in [0,1]$, then $F(x) = x^p$ for some p, $1 \leq p < \infty$. Indeed, since N satisfies the Δ_2-condition, F is a continuous convex function and if $a \in]0,1]$, then $\frac{F(ax)}{F(a)} = \lim_{t\to\infty} \frac{N(atx)}{N(at)} = F(x)$ $x \in [0,1]$, so $F(ax) = F(a)F(x)$; it is wellknown that under these circumstances F has the properties stated.

We can now show the following theorem:

2.5 Theorem

Let $L_M(0,\infty)$ be an reflexive Orlicz space so that for some $p \neq 2$, $$\lim_{t\to\infty} \frac{M(tx)}{M(t)} = x^p, \quad x \in [0,1]$$

If one of the two conditions

(i) $2 \notin [\alpha_M, \beta_M]$ and $L_M(0,\infty)$ is isomorphic to an Orlicz function space on $[0,1]$.

(ii) $L_M(0,\infty)$ is isomorphic to a symmetric function space on $[0,1]$ and either $\max(p,\beta_M) < 2$ or $\min(\alpha_M,p) > 2$ holds, then M is equivalent to x^p.

Proof.

We shall suppose that $p > 2$; the other case will then follow by duality.

By theorem 2.4 both (i) and (ii) imply that $L_M(0,\infty)$ is isomorphic to $L_M(0,1)$ and $\alpha_M = \beta_M = p$, so let $T: L_M(0,\infty) \to L_M(0,1)$ be an isomorphism onto and put $K = \|T\| \|T^{-1}\|$.

Let now $a \geq 1$ be an arbitrary number and let (A_n) be a sequence of mutually disjoint measurable subsets of $[0,\infty[$ with $\lambda(A_n) = M(a)^{-1}$ for all n , and define $f_n = a \cdot 1_{A_n}$, $g_n = Tf_n$ for all $n \in \mathbb{N}$. (f_n) is clearly isometrically equivalent to the unit vector basis of the Orlicz sequence space l_{M_a} , where $M_a(x) = M(a)^{-1}M(ax)$, $x \in [0,1]$, and hence (g_n) is K - equivalent to it.

It now follows from our assumptions that for every $\varepsilon > 0$ we have $(g_n) \not\subseteq A_M^{\varepsilon}(0,1)$; indeed suppose that for some $\varepsilon > 0$ $\lambda\{t \in [0,1] \mid |g_n(t)| \geq \varepsilon\|g_n\|\} \geq \varepsilon$ for all n . By our assumptions on M the formal identity map $I: L_M(0,1) \to L_2(0,1)$ is continuous, so if $\sum_{n=1}^{\infty} t_n g_n$ is convergent, then $\sum_n t_n I(g_n)$ converges unconditionally in $L_2(0,1)$ and hence $\sum_n |t_n|^2 \|I(g_n)\|_2^2 < \infty$, but since $\|I(g_n)\|_2^2 \geq \varepsilon^3 \|g_n\|_M \geq \varepsilon^3 K^{-1}$, this implies that $\sum_n |t_n|^2 < \infty$. On the other hand (g_n) is equivalent to the unit vector basis of l_M , and since $\sum|t_n|^2 < \infty \Rightarrow \sum_n M(|t_n|) < \infty$, we would have that M is equivalent to x^2 at 0 , contradiction.

Since $C_M^{\infty} = \{x^p\}$ it now follows from the proof of proposition 3 in [8] and the results in [3] that if (ε_k) is a sequence tending sufficiently fast to 0 , then there are elements $g_{n_k} \notin A_M^{\varepsilon_k}(0,1)$, so that (g_{n_k}) is a basic sequence, 2-equivalent to the unit vector basis of l_p .

Since (f_n) is isometrically equivalent to each of its subsequences it follows that (f_n) (and hence the unit vector basis of l_{M_a}) is 2K-equivalent to the unit vector basis of l_p. By the Δ_2-property of M, this implies that there is a constant K_1 independent on a, so that

$$(1) \qquad K_1^{-1}x^p \leq M(a)^{-1}M(ax) \leq K_1 x^p \qquad x \in [0,1] .$$

For $a = 1$ we get that M is equivalent to x^p at 0, and if we put $x = a^{-1}$ in (1), we obtain that M is equivalent to x^p at ∞.

q.e.d.

We are also able to prove

<u>2.6 Theorem</u>.

Let M be an Orlicz function, so that $\lim\limits_{t\to\infty} \frac{M(tx)}{M(t)} = x^p$

$x \in [0,1]$ for some p, $1 \leq p < \infty$.

If $L_M(0,\infty)$ is lattice isomorphic to $L_M(0,1)$, then

M is equivalent to x^p.

<u>Proof.</u>

This can be proved as theorem 2.5. Indeed the only place, where the conditions on M were used there, was in the technique involving the sets A_M^ε, and we need not use this argument under the assumptions of the present theorem, since a lattice isomorphism maps disjoint functions onto disjoint functions.

q.e.d.

The following theorem shows that if $L_M(0,\infty)$ is isomorphic to $L_M(0,1)$ and M is not equivalent to x^p, then there are no "nice" isomorphism between the spaces.

<u>2.7 Theorem</u>

Let $d(L_M(0,\infty), L_M(0,1)) < \infty$. If there is a lattice isomorphism T of $L_M(0,1)$ onto $L_M(0,\infty)$ and an Orlicz function N equivalent to M, so that

$$\text{(i)} \qquad \int_0^\infty N(|Tf|)d\lambda = \int_0^1 N(|f|)d\lambda \qquad f \in L_M(0,1).$$

then there is a p, $1 \leq p < \infty$ with M equivalent to x^p.

<u>Proof</u>

Let $\mathcal{B}_1$ (respectively $\mathcal{B}_\infty$) denote the set of equivalence classes of the Borel sets in $[0,1]$ (respectively $[0,\infty[$.), and let us define a map $\varphi : \mathcal{B}_\infty \to \mathcal{B}_1$ by

$$\text{(1)} \qquad \varphi(A) = \operatorname{supp} T^{-1}(1_A) \qquad A \in \mathcal{B}_\infty.$$

Since T is a lattice isomorphism it is easy to see that φ satisfies the following conditions

$$\text{(2)} \qquad \mu(\varphi([0,\infty[)) = 1.$$

$$\text{(3)} \qquad \varphi([0,\infty[\setminus A) = \varphi([0,\infty[)\setminus \varphi(A) \qquad A \in \mathcal{B}_\infty.$$

$$\text{(4)} \qquad \varphi(\bigcup_n A_n) = \bigcup \varphi(A_n) \quad \text{for mutually disjoint } A_n \in \mathcal{B}_\infty$$

It is wellknown that under these circumstances φ induces a linear map φ^o from the space of measurable functions on $[0,1]$ onto the space of measurable functions on $[0,\infty[$, so that $\varphi^o(1_A) = 1_{\varphi^{-1}(A)}$ $A \in \mathcal{B}_1$.

Let now f be a simple function on $[0,1]$, say $f = \sum_{i=1}^n a_i 1_{A_i}$, where $\bigcup_{i=1}^{n} A_i = [0,1]$, $A_i \cap A_j = \emptyset$, $i \neq j$.

For every $i \leq n$ the support of the function $f - a_i 1_{[0,1]}$ is disjoint from A_i and hence by the definition of φ we get that $\operatorname{supp}(Tf - a_i \cdot 1_{[0,1]})$ is disjoint from $\varphi^{-1}(A_i)$, hence for all $i \leq n$:

$$(5) \qquad Tf = a_i T(1) 1_{\varphi^{-1}(A_i)} = T(1)\varphi^{o} f \quad \text{on } A_i$$

and summing over i we obtain (5) everywhere. By continuity of T we get that (5) holds for all functions in $L_M(0,1)$.

If we put $u = T(1)$, then we get for all $r \in [0,\infty[$ and all $A \in \mathcal{B}_\infty$

$$N(r)\varphi^{-1}(\lambda)(A) = \int_0^1 N(r 1_{\varphi(A)}) d\lambda = \int_A N(ru) d\lambda .$$

By the uniqueness of Radon-Nikodym derivations and the continuity of N, we get for a suitable set $A \subseteq [0,\infty[$ $\lambda(A) = 0$:

$$(6) \qquad N(u(t)r) = N(r)N(u(t)) \qquad r \in [0,\infty[\qquad t \in [0,\infty[\setminus A .$$

Since $u \in L_M(0,\infty)$ there is a t_o, so that $o < a = u(t_o) \neq 1$, clearly

$$(7) \qquad N(a^n x) = N(a^n)N(x) \qquad \text{for all } x \geq 0 \text{ and } n = 0, \pm 1, \pm 2, \ldots$$

From (7) it follows immediately that there exists a constant $K \geq 0$ so that $K^{-1}N(t)N(x) \leq N(tx) \leq KN(t)N(x)$ for all $t,x \geq 0$, and therefore there is a p, so that $N \sim x^p$, and the same holds for M.

q.e.d.

Let us now return again to the general case.

If M is an Orlicz function and $s \in [0,1]$, then we put $M_s(x) = \frac{M(sx)}{M(s)}$ $x \in \mathbb{R}$. Since M_s is equivalent to M, we get of course that $d(L_M(0,1), L_{M_s}(0,1)) < \infty$, but it is easy to see that in general this distance depends on s , and the question is then, when it is uniformly bounded in s from above. We have the following theorem.

2.8 Theorem.

Let M be an Orlicz function so that $L_M(0,\infty)$ is isomorphic to $L_M(0,1)$. Then there is a constant K , so that $d(L_M(0,1), L_{M_s}(0,1)) \leq K$ for all $s \in]0,1]$.

Proof.

Let us first recall that if $\bar{X}$ and Y are two Banach spaces, then the space $(\bar{X} \oplus Y)_{c_0}$ is the space $\bar{X} \times Y$ equipped with the norm $\|(x,y)\| = \max(\|x\|,\|y\|)$ for $(x,y) \in \bar{X} \times Y$.

If $a \geq 1$ is arbitrary, then it is easy to see that $L_M(a,\infty)$ is isometric to $L_M(0,\infty)$ and that

(1) $L_M(0,a) = L_M(0,1) \oplus L_M(1,a)$.

(2) $L_M(0,\infty) = L_M(0,a) \oplus L_M(a,\infty)$.

Letting "$\sim$" denote "isomorphic to" we have the following scheme:

(3)
$$\begin{aligned} L_M(0,1) &\sim L_M(0,\infty) = L_M(0,a) \oplus L_M(a,\infty) \sim \\ &L_M(0,a) \oplus L_M(0,\infty) \sim (L_M(0,a) \oplus L_M(0,1)) \sim \\ &((L_M(0,1) \oplus L_M(1,a)) \oplus L_M(0,1)) \sim \\ &(L_M(0,1) \oplus L_M(1,a)) \sim L_M(0,a) . \end{aligned}$$

Using the isomorphisms in (3) to compute $d(L_M(0,a),L_M(0,1))$ we find that there is a constant K independent of a , so that

(4) $d(L_M(0,a), L_M(0,1)) \leq K$. for all $a \geq 1$.

Since $L_M(0,a)$ is readily seen to be isometric to $L_{M_s}(0,1)$ with $s = M^{-1}(a^{-1})$, the conclusion of the theorem follows directly from (4).

q.e.d.

Theorem 2.8 naturally leads to the following conjecture.

2.9 Conjecture.

Let M be an Orlicz function and let $\{N_\alpha | \alpha \in I\}$ be a family of Orlicz functions, each having the same Δ_2-constant as M. If $\sup_\alpha d(L_M(0,1), L_{N_\alpha}(0,1)) < \infty$, then there is a constant K independent on α so that

$$K^{-1}N_\alpha(x) \leq M(x) \leq KN_\alpha(x) \qquad x \geq 1 .$$

If this conjecture is answered positively, then it would follow from the theorems 2.4 and 2.8 that under the conditions in 2.4 an Orlicz space $L_M(0,\infty)$ is isomorphic to a symmetric function space on $(0,1)$ (an Orlicz space on $(0,1)$ under 2.4 ii) if and only if $M \sim x^p$ for some $p, 1 < p < \infty$. Indeed, M and the family $\{M_s | s \in]0,1]\}$ will satisfy the conditions of 2.9, and hence there would be a constant K so that

$$(*) \qquad K^{-1}M(x)M(s) \leq M(sx) \leq KM(s)M(x) \qquad 0 < s < 1 \quad x \geq 1$$

and it is wellknown that an Orlicz function satisfying $(*)$ is equivalent to x^p for some $p 1 \leq p < \infty$.

In theorem 4 of [8] it is shown that if M and N are Orlicz functions with $1,2 \notin [\alpha_M^\infty, \beta_M^\infty]$, so that $L_M(0,1)$ is isomorphic to $L_N(0,1)$, then M is equivalent to N at ∞. One could hope that the proof of this (similar to our proof of 2.3) would show that the equivalence constant between M and N only depends on $d(L_M(0,1), L_N(0,1))$ and the Δ_2-constants of M and N, and hence answer conjecture 2.9. Unfortunately this is not the case, indeed, the main point in the proof (as in 2.3 here) is to construct a "characteristic" sequence in $L_N(0,1)$ with the aid of the Dacunha-Castelle inequality (formula (3) in 2.3), whose image in $L_M(0,1)$ is contained in A_M^{ε} for a suitable ε. While it is easy to check that the constant in the above mentioned inequality only depends on the Δ_2-constant of M, the ε obviously depends on the chosen sequence, and hence the finally computed constant will also do it. If we apply this technique to the setting of 2.9 we will get a family $\{\varepsilon_\alpha\}$ of numbers and it seems impossible to show that this family is bounded away from 0, even in the case, where the family is $\{M_s | 0 < s \le 1\}$.

As it is seen, it seems as the Kadec-Pełczyński "A_M^{ε}-technique" is too weak to give an answer to conjecture 2.9. Recently, however, Pełczyński and Rosenthal [11] have obtained finite dimensional versions of the results of Kadec and Pełczyński [3] on L_p-spaces, and it is likely that the methods used here would be strong enough to salve conjectures like 2.9, if they can be carried over to the Orlicz space case (but that this is possible, is not so straight forward as in case of the Kadec-Pełczyński technique, and so far we have been unable to do it).

3. Some additional remarks and open problems.

The main problems left open in this paper are of course centered around the question, whether or not theorem 2.5 can be generalized to the class consisting of all Orlicz functions.

Of other problems on the Orlicz spaces $L_M(0,\infty)$ we can mention:

3.1 Problem.

Let M and N be Orlicz functions, so that $1,2 \notin [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty]$. Suppose that $L_M(0,\infty)$ is isomorphic to $L_N(0,\infty)$. Are M and N equivalent?
What is the situation if $1,2 \in [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty]$?

Applying theorem 2.3 and the technique of the proof of theorem 2.4 it is not difficult to show that if M and N satisfies the conditions of the first question in 3.1, then M and N are equivalent at ∞, so the hard task is to decide how the Orlicz functions behave close to 0. Here one should perhaps keep in mind that while the problem similar to 3.1 in the case $[0,1]$ is solved positively by theorem 4 of [8], then there are also examples of non-equivalent Orlicz functions M and N, so that l_M and l_M are isomorphic. The second question and even the similar one for the case of $[0,1]$ are both wide open. The reason for dividing problem 3.1 into two cases is that in the first case it will be possible to use an reflexivity argument together with the important theorem 2.3, which cannot be generalized, at least not to the case, where $\beta_M^\infty \geq 2$, since $L_2(0,1)$ is isomorphic to a subspace of every Orlicz space $L_M(0,\infty)$ (via the Rademacher functions on $[0,1]$ and the Khinchin inequality). We believe however that 3.1 can be proved in the affirmative without using 2.3.

Similar remarks on the role of theorem 2.3 can be applied to the following problem:

<u>3.2 Problem.</u>

Can the restriction $1,2 \notin [\alpha_M,\beta_M] \cup [\alpha_M^\infty,\beta_M^\infty]$ be removed in theorem 2.4 (ii)?

Is (i) of theorem 2.4 true under the same conditions on M as in (ii)?

The second question in 3.2 will follow from

<u>3.3 Problem.</u>

If $L_M(0,\infty)$ is isomorphic to a $L_p(0,1)$, is $L_p(0,1)$ then isomorphic to an Orlicz space on $[0,1]$.

We strongly believe that 3.3 can be answered in the affirmative by a proof, which does not involve imbeddings into L_p-spaces as corollary 2.2 does.

Let us finally give a few important examples of Orlicz functions, different from the x^p-functions, belonging to the class considered in 2.5 (and hence of Orlicz spaces on $(0,\infty)$, which are not isomorphic to symmetric function spaces on $(0,1)$.).

Example 3.4.

Let $1 < p \neq 2$ and let N be an arbitrary Orlicz function, whose interval $[\alpha_N, \beta_N]$ is to "the same side" of 2 as p. If we put $M(x) = x^p \log x$ in a suitable neighbourhood of ∞, $M(x) = N(x)$ in a suitable neighbourhood of 0 and modify M in between to be convex, then M belongs to the class of 2.5. The same is true in the case, where we put $M(x) = x^p/\log x$ in a neighbourhood of ∞.

Example 3.5.

Let $K \subseteq]1,\infty[$ be an infinite compact set so that $p = \sup K \neq 2$, let μ be a probability measure on K with support K and so that $\mu\{p\} = 0$, and let N be an Orlicz function with $[\alpha_N, \beta_N]$ to "the same side" of 2 as p. Define:

$$M(x) = \int_K x^s d\mu(S) \quad \text{in a suitable neighbourhood of } \infty .$$

$$M(x) = N(x) \quad \text{in a suitable neighbourhood of } 0 .$$

Since it is easily shown that $\lim\limits_{t\to\infty} \frac{M(tx)}{M(t)} = x^p$ for all $x \in [0,1]$, M belongs to the class of Orlicz functions considered in 2.5, and since $\mu\{p\} = 0$, M is not equivalent to x^p at ∞.

Other examples can be found in [6], [7] and [8].

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